Parameter-free Locally Accelerated Conditional Gradients

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Abstract

Conditional gradient (CG) methods are the algorithms of choice for constrained optimization when projections are computationally prohibitive but linear optimization over the constraint set remains possible. Unlike in projection-based methods, globally accelerated convergence rates are in general unattainable for CG. One can achieve *local acceleration* with knowledge of the smoothness and strong convexity parameters of the function [1]. We remove this limitation by introducing the Parameter-Free Locally accelerated CG (PF-LaCG) algorithm.

Motivation

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) \qquad (1)$$

Goal is L-smooth μ-strongly convex optimization over a polytope $\mathcal{X}$ with **First Order Oracles** (FOO) and **Linear Minimization Oracle** (LMO). Focus on the *Conditional Gradients* (CG) algorithm [2, 3], and its variants, such as the *Away-step Frank-Wolfe* (AFW) algorithm.

Convergence rate of CG variants

The number of steps T required to reach an ϵ-optimal solution to Problem (1) [4]:

$$T = O\left(\frac{L}{\mu}\left(\frac{D}{\delta}\right)^2 \log\frac{1}{\epsilon}\right),$$

where D and δ are the diameter and pyramidal width of $\mathcal{X}$, and D/δ is dimension-dependent.

The rates of first-order optimal projection-based methods [5]: 1) Depend on $\sqrt{L/\mu}$ and 2) Do not depend on the dimension.

These rates cannot be achieved *globally* [6] with the LMO, but they can be achieved locally if we know L and μ [1]:

*Can CG achieve these rates locally without knowing L and μ? **Yes!***

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Our contributions are:

1. Parameter-free Locally-accelerated Conditional Gradient (PF-LaCG) algorithm.
2. Near-optimal and parameter-free accelerated algorithm (ACC) with inexact projections.

We achieve local acceleration by coupling the AFW and ACC algorithm and restarting when an **upper bound on the primal gap** is halved:

$$w(\mathbf{x},\mathcal{S}) \stackrel{\text{def}}{=} \max_{\mathbf{u}\in\mathcal{X},\mathbf{v}\in\mathcal{S}} \langle \nabla f(\mathbf{x}), \mathbf{u}-\mathbf{v}\rangle .$$

where $\mathcal{S}$ is a proper support. This allow us to maintain a computable global measure of optimality without knowing L and μ and couple the AFW and ACC algorithms while guaranteeing monotonic progress in $w(\mathbf{x},\mathcal{S})$.

Convergence rate of ACC

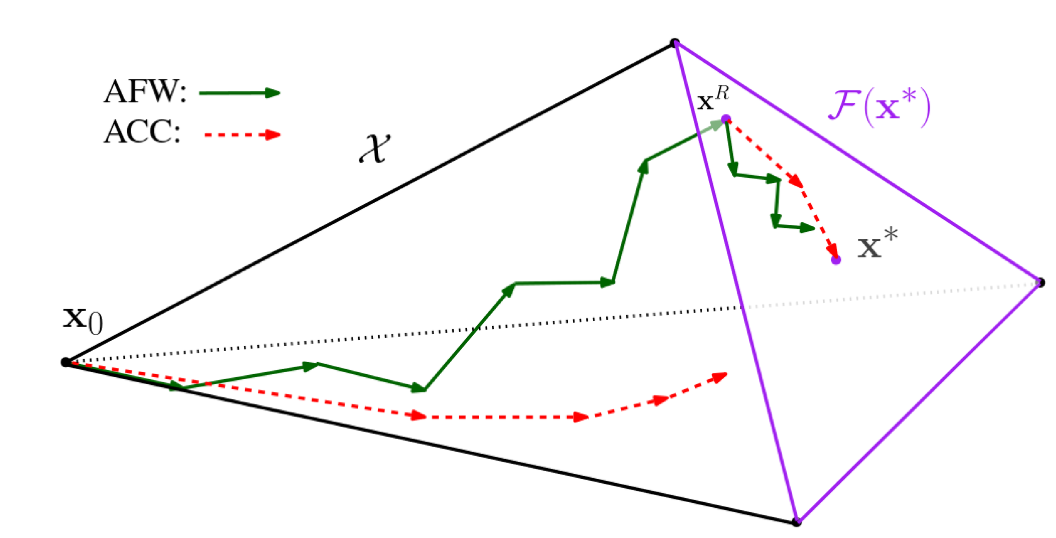
Let $C \subseteq \mathbb{R}^n$ be a closed convex set, such that $\mathbf{x}^* \subset C$. Then running the ACC with properly initialized parameters over C outputs a point with dual optimality gap smaller than ϵ with a total number of

$$K = O\left(\sqrt{\frac{L}{\mu}}\log\left(\frac{L}{\mu}\right)\log\left(\frac{L}{\mu\epsilon}\right)\right)$$

queries to the FOO for f and an inexact and efficiently computable projection oracle for C, without knowledge of L or μ.

Algorithm Main Ideas

1. PF-LaCG runs AFW and ACC in parallel, and restarts every time AFW halves $w(\mathbf{x},\mathcal{S})$. After every restart choose point with lower value of $w(\mathbf{x},\mathcal{S})$ and potentially update active set of ACC
2. After a finite number of iterations independent of ϵ, the active set of AFW contains $\mathbf{x}^*$ **and** ACC converges to the optimum at an accelerated rate



Convergence rate of PF-LaCG

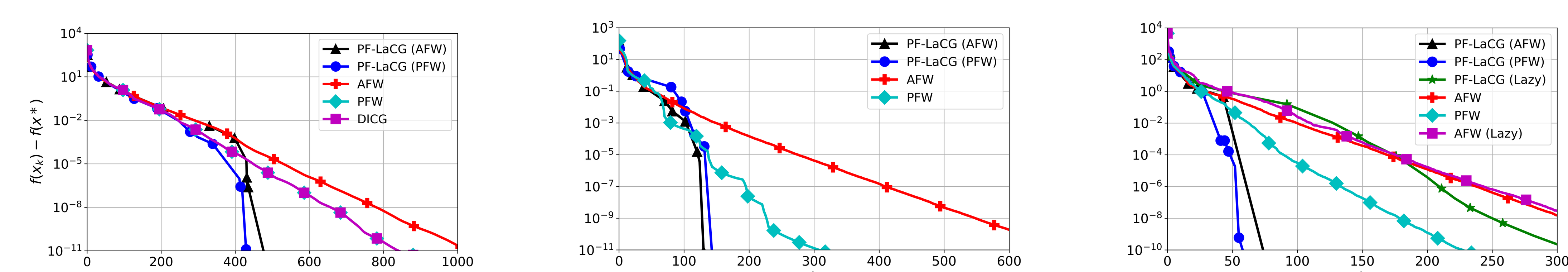
Let f be L-smooth and μ-strongly convex. The number of calls to FOO and LMO required to reach an ϵ-optimal solution, measured in terms of $w(\mathbf{x},\mathcal{S})$, to the minimization problem satisfies:

$$T = \min\left\{\underbrace{O\left(\frac{LD^2}{\mu\delta^2}\log\frac{1}{\epsilon}\right)}_{\text{AFW bound}},\ \underbrace{K}_{\text{Burn-in}} + \underbrace{O\left(\sqrt{\frac{L}{\mu}}\log\left(\frac{L}{\mu}\right)\log\left(\frac{LD}{\mu\delta}\right)\log\frac{1}{\epsilon}\right)}_{\text{Locally-accelerated convergence}}\right\},$$

where K is a constant that is independent of ϵ.

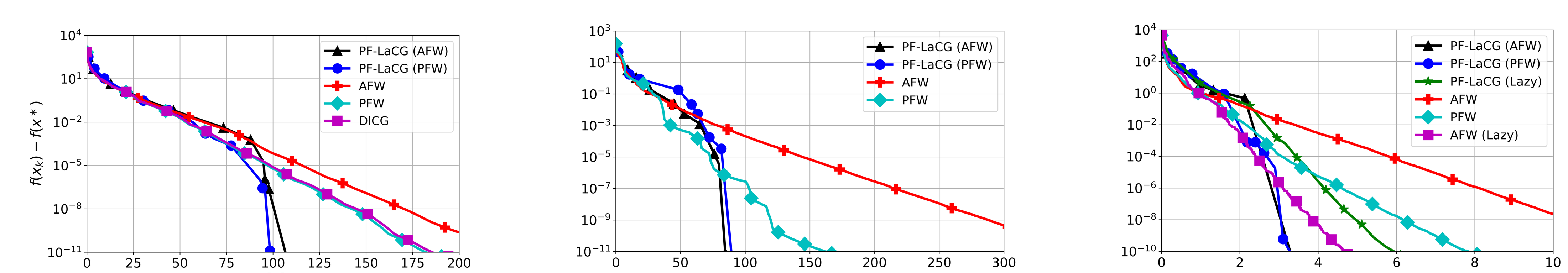
PF-LaCG achieves parameter-free local acceleration (optimal up to poly-log factors)

Computational Results



(b) Probability simplex problem (c) Structured lasso problem (d) Structured Birkhoff problem

Figure 1: Performance w.r.t. iteration count: Algorithm comparison for a strongly-convex and smooth quadratic problem over the probability simplex (b), a structured lasso feasible region (c), and a structured Birkhoff polytope domain (d) with respect to iteration count.



(a) Probability simplex problem (b) Structured lasso problem (c) Structured Birkhoff problem

Figure 2: Performance w.r.t. wall-clock time: Algorithm comparison for a strongly-convex and smooth quadratic problem over the probability simplex (a), a structured lasso feasible region (b), and a structured Birkhoff polytope domain (c) with respect to wall-clock time.

Paper: https://arxiv.org/pdf/2102.06806.pdf **Code**: https://github.com/ericlincc/Parameter-free-LaCG

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